

On finiteness of the sum of negative eigenvalues of Schrödinger operators

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Abstract

We prove conditions on potentials V which imply that the sum of the negative eigenvalues of the Schrödinger operator $-\Delta + V$ is finite. We use a method for bounding eigenvalues based on estimates of the Hilbert-Schmidt norm of semigroup differences and on complex analysis.

1 Introduction

A basic theme in the theory of Schrödinger operators $H = -\Delta + V$ is to relate the properties of the potential V to properties of the set of eigenvalues of H . In this paper we prove conditions on the potential which are sufficient in order that the sum of negative eigenvalues of the Schrödinger operator be finite:

$$\sum_{\lambda \in \sigma^-(H)} |\lambda| < \infty, \quad (1)$$

where $\sigma^-(H) = \sigma(H) \cap (-\infty, 0)$, the negative part of the spectrum of H .

Our main result is the following

Theorem 1 *Assume $d \geq 4$. Let $V : \mathbb{R}^d \rightarrow \mathbb{R}$ be a Kato potential, and assume that $V_- = \min(V, 0)$ satisfies, for some $c > 0$,*

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{-c|w-w'|^2} |V_-(w)| |V_-(w')| dw dw' < \infty. \quad (2)$$

and also

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(i) If $d = 4$ then

$$\iint_{|w-w'|<1} \log\left(\frac{1}{|w-w'|}\right) |V_-(w)| |V_-(w')| dw' dw < \infty. \quad (3)$$

(ii) If $d \geq 5$ then

$$\iint_{|w-w'|<1} \frac{|V_-(w)| |V_-(w')|}{|w-w'|^{d-4}} dw' dw < \infty. \quad (4)$$

Then (1) holds.

From the above Theorem we derive the following L^p -conditions on V for (1) to hold:

Corollary 1 Assume $d \geq 4$ and V is Kato, and $V_- \in L^p$, where $p \in [\frac{2d}{d+4}, 2]$. Then (1) holds.

Corollary 2 If $d \geq 4$, V is Kato and $V_- \in L^1$, then (1) holds.

It is interesting to compare these results with those that can be obtained from the Lieb-Thirring inequalities, which also give some L^p -conditions implying (1). The Lieb-Thirring inequalities [5, 3]

$$\sum_{\lambda \in \sigma^-(H)} |\lambda|^\gamma \leq C_{d,\gamma} \int_{\mathbb{R}^d} |V_-(x)|^{\frac{d}{2}+\gamma} dx, \quad (5)$$

hold for $\gamma \geq \frac{1}{2}$ when $d = 1$, for $\gamma > 0$ when $d = 2$, and for $\gamma \geq 0$ when $d \geq 3$. Since finiteness of the left-hand side of (5) for any $\gamma \leq 1$ implies (1), we get the following sufficient conditions for (1) to hold:

- (1) $d = 1$, $V_- \in L^p$, where $p \in [1, \frac{3}{2}]$.
- (2) $d = 2$, $V_- \in L^p$, where $p \in (1, 2]$.
- (3) $d \geq 3$, $V_- \in L^p$, where $p \in [\frac{d}{2}, \frac{d}{2} + 1]$.

Comparing with our Corollaries 1, 2, we see that in the case $d = 4$ the Lieb-Thirring inequalities give (1) when $p \in [2, 3]$, while Corollary 1 gives the range of values $p \in [1, 2]$, so together we have the range $p \in [1, 3]$. In the case $d \geq 5$ the ranges of values of p for which (1) holds given by Corollary 1 are *disjoint* from the range of values given by the Lieb-Thirring inequalities. We also note that the result Corollary 2 does not follow from the Lieb-Thirring results.

An immediate question is whether the results of Corollaries 1,2 hold in dimensions 1,2,3, that is whether the restriction $d \geq 4$ that we impose is an artifact of our method of proof or a reflection of the actual situation. In fact we can

construct a counterexample showing that the result of Corollary 1 is *not* true when $d = 1$. Considering a potential of the form $V(x) = -(1 + |x|)^{-\alpha}$, one has $V \in L^2(\mathbb{R})$ when $\alpha > \frac{1}{2}$. A WKB approximation shows that, when $\alpha \in (0, 2)$, the n -th eigenvalue satisfies $\lambda_n \sim n^{-\frac{2\alpha}{2-\alpha}}$, so that when $\alpha < \frac{2}{3}$ the sum of the eigenvalues diverges. Thus, for $\alpha \in (\frac{1}{2}, \frac{2}{3})$ we have that $V \in L^2(\mathbb{R})$, yet the sum of eigenvalues diverges. On the other in the case of Corollary 2, the Lieb-Thirring results show that it is also valid for $d = 1$. We do not know whether Corollaries 1,2 are valid in dimensions $d = 2, 3$.

The technique we use for the proof of Theorem 1 is a considerable refinement of ideas we introduced in [2]. There we developed a method, based on the Jensen identity of complex analysis, to bound the moments (sums of powers) of the negative eigenvalues of a self-adjoint operator B on a complex Hilbert space \mathcal{H} , assuming that there is a self-adjoint operator A with $\sigma(A) \subset [0, \infty)$, such that the semigroup difference $D_t = e^{-tB} - e^{-tA}$ is a trace class or Hilbert-Schmidt operator. We obtained some general ‘abstract’ results bounding the moments of eigenvalues. Applied to Schrödinger operators, these results implied that, under appropriate conditions on the potential, the moment sum on the left-hand side of (5) is finite for $\gamma > 2$. Theorem 1, which corresponds to the case $\gamma = 1$, is proven using the same method, but with the difference that by restricting ourselves to Schrödinger operators rather than general selfadjoint operators, we are able to improve the estimates in such a way that the stronger result is proven.

We note that from the proof of Theorem 1 one can extract explicit bounds for the sum of negative eigenvalues, in terms of V_- (Kato norms and the quantities given in (2),(3),(4)). However these expressions are rather cumbersome, so we have decided to concentrate on the more ‘qualitative’ aspect of the results.

In the following section we recall the method developed in [2]. In Section 3 we apply the method to obtain the proof of Theorem 1.

2 The Jensen formula and eigenvalues

In this section we recall the technique developed in [2]. Assume that A, B are self-adjoint operators in a complex Hilbert space \mathcal{H} , with $\sigma(A) \subset [0, \infty)$, B semibounded from below, and that the difference of semigroups $D_t = e^{-tB} - e^{-tA}$ is Hilbert-Schmidt, for some $t > 0$. These assumptions imply, by Weyl’s Theorem, that $\sigma_{ess}(B) = \sigma_{ess}(A) \subset [0, \infty)$ so the negative part of the spectrum $\sigma^-(B)$ consists only of eigenvalues, which can only accumulate at 0.

We define the operator-valued function

$$F(z) = z[I - ze^{-tA}]^{-1}D_t, \quad (6)$$

on $\Omega = \{z \in \mathbb{C} \mid |z| < 1\}$. Note that the assumption $\sigma(A) \subset [0, \infty)$ implies that

the inverse $[I - ze^{-tA}]^{-1}$ is well-defined. We have the identity

$$[I - ze^{-A}]^{-1}[I - ze^{-B}] = I - F(z),$$

which implies, for $\lambda < 0$,

$$\lambda \in \sigma^-(B) \Leftrightarrow 1 \in \sigma(F(e^\lambda)). \quad (7)$$

The assumption that D_t is Hilbert-Schmidt implies that $F(z)$ is Hilbert-Schmidt, and we can define the holomorphic function $h(z)$ in $|z| < 1$ by

$$h(z) = \text{Det}_2(I - F(z)), \quad (8)$$

where Det_2 denotes the regularized determinant defined for Hilbert-Schmidt perturbations of the identity (see e.g. [8]).

From (7) we have, for $\lambda < 0$,

$$\lambda \in \sigma^-(B) \Leftrightarrow h(e^\lambda) = 0, \quad (9)$$

and moreover the multiplicity of λ as an eigenvalue of B coincides with multiplicity of e^λ as a zero of h .

We now recall the Jensen identity from complex analysis (see e.g. [6], p. 307).

Lemma 1 *Let Ω be the open unit disk. Let $h : \Omega \rightarrow \mathbb{C}$ be a holomorphic function, and assume $h(0) = 1$. Then, for $0 \leq r < 1$,*

$$\frac{1}{2\pi} \int_0^{2\pi} \log(|h(re^{i\theta})|) d\theta = \log \left(\prod_{|z| \leq r, h(z)=0} \frac{r}{|z|} \right).$$

A variation on a particular case of the results of [2] which we will use here is:

Theorem 2 *Let A, B be self-adjoint in a complex Hilbert space \mathcal{H} , with $\sigma(A) \subset [0, \infty)$. Assume that, for some $t > 0$, $D_t = e^{-tB} - e^{-tA}$ is Hilbert-Schmidt. Then, defining h by (8), we have*

$$\sum_{\lambda \in \sigma^-(B)} |\lambda| = \frac{1}{t} \lim_{r \rightarrow 1-} \frac{1}{2\pi} \int_0^{2\pi} \log(|h(re^{i\theta})|) d\theta. \quad (10)$$

Proof: By Jensen's identity and (9) we have

$$\begin{aligned} & \lim_{r \rightarrow 1-} \frac{1}{2\pi} \int_0^{2\pi} \log(|h(re^{i\theta})|) d\theta = \log \left(\prod_{|z| < 1, h(z)=0} \frac{1}{|z|} \right) \\ &= \log \left(\prod_{\lambda \in \sigma^-(B)} \frac{1}{e^\lambda} \right) = \sum_{\lambda \in \sigma^-(B)} |\lambda|. \end{aligned}$$

■

Theorem 2 shows that one can bound the sum of the negative eigenvalues by bounding the function h , and this is our task now.

Let us first note that, by the general inequality

$$|\text{Det}_2(I - T)| \leq e^{\frac{1}{2}\|T\|_{HS}^2}$$

for Hilbert-Schmidt operators T , we have

$$\log(|h(z)|) \leq \frac{1}{2}\|F(z)\|_{HS}^2, \quad (11)$$

so that we can bound $h(z)$ by bounding the Hilbert-Schmidt norm of $F(z)$. To do this, one can - and this is what was done in [2] - use the inequality

$$\|F(z)\|_{HS} \leq |z| \| [I - ze^{-tA}]^{-1} \| \|D_t\|_{HS}, \quad (12)$$

where the norm $\|[I - ze^{-tA}]^{-1}\|$ is the regular operator norm, which can in turn be bounded in terms of the inverse distance of the spectrum of $I - ze^{-tA}$ to 0, using the assumption that $\sigma(A) \subset [0, \infty)$. In this way we obtain the general results of [2].

The observation at the basis of this work is that, when the operators A, B are Schrödinger operators, the bound on $\|F(z)\|_{HS}$ obtained by using (12) is not optimal, and one can obtain better bounds in the Schrödinger case by *not* separating the estimation into two parts as in (12). For example the bounds we obtain show that when $d \geq 5$, the function $h(z)$ is *uniformly* bounded in the unit disk $|z| < 1$, whereas the bound obtained by using (12) goes to $+\infty$ as $z \rightarrow 1$. These improved bounds lead, through Theorem 2, to improved bounds on the sum of the negative eigenvalues of B .

3 Proofs

Recall that the potential $V : \mathbb{R}^d \rightarrow \mathbb{R}$ is said to belong to the class $K(\mathbb{R}^d)$ if

$$\lim_{t \rightarrow 0} \sup_{x \in \mathbb{R}^d} \int_0^t (e^{\eta \Delta} |V|)(x) d\eta = 0.$$

We note that when $d \geq 3$, a necessary and sufficient condition for $V \in K(\mathbb{R}^d)$ is that

$$\lim_{\alpha \rightarrow 0} \left[\sup_{x \in \mathbb{R}^d} \int_{|y-x| \leq \alpha} \frac{|V(y)|}{|y-x|^{d-2}} dy \right] = 0. \quad (13)$$

We recall also that when $d \geq 3$, a *sufficient* condition for $V \in K(\mathbb{R}^d)$ (see [7]) is that V is uniformly-locally in L^p for some $p > \frac{d}{2}$, that is

$$\sup_{x \in \mathbb{R}^d} \int_{|y-x| \leq 1} |V(x)|^p dx < \infty.$$

V is said to belong to class $K^{loc}(\mathbb{R}^d)$ if $\chi_Q V \in K(\mathbb{R}^d)$ for any ball $Q \subset \mathbb{R}^d$, where χ_Q denotes the characteristic function of Q . V is said to be a Kato potential if $V_- = \min(V, 0) \in K(\mathbb{R}^d)$ and $V_+ = \max(V, 0) \in K^{loc}(\mathbb{R}^d)$.

By the min-max principle, the eigenvalues of $-\Delta + V_-$ are smaller than or equal to the corresponding eigenvalues of $-\Delta + V$, and therefore we have

$$\sum_{\lambda \in \sigma^-(-\Delta + V)} |\lambda| \leq \sum_{\lambda \in \sigma^-(-\Delta + V_-)} |\lambda|, \quad (14)$$

so that to prove Theorem 1 it suffices to show that the right-hand side of (14) is finite. We shall therefore take $A = H_0 = -\Delta$, $B = H_0 + V_-$, so that

$$D_t = e^{-t(H_0 + V_-)} - e^{-tH_0}.$$

We recall some fundamental facts about Schrödinger semigroups (see e.g. [1, 7]), which will be needed below:

Lemma 2 *If $V_- \in K(\mathbb{R}^d)$ then the Schrödinger semigroup $e^{-t(H_0 + V_-)} : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$ ($t \geq 0$) is well defined, and moreover we have, for all $t > 0$,*

$$\begin{aligned} \|e^{-t(H_0 + V_-)}\|_{L^1, L^\infty} &< \infty, \\ \sup_{s \in [0, t]} \|e^{-s(H_0 + V_-)}\|_{L^\infty, L^\infty} &< \infty. \end{aligned}$$

As explained in the previous section, our task is to bound the norm $\|F(z)\|_{HS}$, where $F(z)$ is given by (6).

We define the operator-valued function $G(z)$, $|z| < 1$, by

$$G(z) = ze^{-tA}[I - ze^{-tA}]^{-1}.$$

It is easily checked that

$$[I - ze^{-tA}]^{-1} = I + G(z),$$

hence

$$F(z) = z[I + G(z)]D_t,$$

so that

$$\|F(z)\|_{HS} \leq |z|[\|D_t\|_{HS} + \|G(z)D_t\|_{HS}] \quad (15)$$

We are going to bound the two terms on the right-hand side of (15).

We divide the required estimates into several steps.

3.1 Some estimates on $G(z)$

Lemma 3 *The operator $G(z)$ can be represented in the form*

$$G(z)f = g_z * f, \quad \forall f \in L^2(\mathbb{R}^d), \quad (16)$$

where $g_z \in L^\infty(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$.

Proof: From the definition of $G(z)$ and the properties of the Fourier transform we have, for $f \in L^2(\mathbb{R}^d)$,

$$\mathfrak{F}(G(z)f) = ze^{-t|\xi|^2} [1 - ze^{-t|\xi|^2}]^{-1} \mathfrak{F}(f),$$

so that if define $g_z : \mathbb{R}^d \rightarrow \mathbb{C}$, for $|z| < 1$, by

$$g_z = z \mathfrak{F}^{-1}(e^{-t|\xi|^2} [1 - ze^{-t|\xi|^2}]^{-1})$$

we get (16). Since $e^{-t|\xi|^2} [1 - ze^{-t|\xi|^2}]^{-1} \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, we have $g_z \in L^\infty(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. ■

We note that while $\|G(z)\|_{L^2, L^2} = \|g_z\|_{L^1} \rightarrow \infty$ when $z \rightarrow 1$, we are going to show - and this is a key technical point for obtaining Theorem 1 - that when $d \geq 5$ the norm $\|g_z\|_{L^2}$ is in fact bounded for $|z| < 1$.

We will denote, for $|z| < 1$,

$$M(z) = \|g_z\|_{L^2}. \quad (17)$$

We will need the following elementary estimates:

Lemma 4 *Assuming $p > 0$, $a < 1$, Let*

$$J_p(a) = \int_1^\infty \frac{(\log(s))^{p-1}}{(s-a)^2} ds. \quad (18)$$

Then:

(a) *For $p = 2$ we have*

$$J_2(a) = O\left(\log\left(\frac{1}{1-a}\right)\right), \quad \text{as } a \rightarrow 1-$$

(b) *For $p > 2$ we have*

$$J_p(a) = O(1), \quad \text{as } a \rightarrow 1-$$

Proof: We write

$$\int_1^\infty \frac{(\log(s))^{p-1}}{(s-a)^2} ds = \int_1^2 \frac{(\log(s))^{p-1}}{(s-a)^2} ds + \int_2^\infty \frac{(\log(s))^{p-1}}{(s-a)^2} ds.$$

The second integral on the right-hand side is obviously finite and bounded independently of $a \in (-\infty, 1)$. We continue estimating the first integral.

Assuming $p \geq 1$, and using the fact that $\log(s) \leq s - 1$ for $s \geq 1$, we have

$$\begin{aligned} & \int_1^2 \frac{(\log(s))^{p-1}}{(s-a)^2} ds \leq \int_1^2 \frac{(s-1)^{p-1}}{(s-a)^2} ds = \int_1^2 \frac{(s-1)^{p-1}}{(s-a)^{3-p}(s-a)^{p-1}} ds \\ & \leq \int_1^2 \frac{(s-1)^{p-1}}{(s-a)^{3-p}(s-1)^{p-1}} ds = \int_1^2 \frac{1}{(s-a)^{3-p}} ds \\ & = \begin{cases} \log\left(\frac{2-a}{1-a}\right) & p = 2 \\ \frac{1}{2-p}[(1-a)^{p-2} - (2-a)^{p-2}] & 1 \leq p \neq 2 \end{cases} \quad \text{as } a \rightarrow 1-. \\ & = \begin{cases} O\left(\log\left(\frac{1}{1-a}\right)\right) & p = 2 \\ O(1) & p > 2 \end{cases} \quad \text{as } a \rightarrow 1-. \end{aligned}$$

■

We now present our main estimate on $M(z)$.

Lemma 5 Define $M(z)$ by (17).

(a) If $d = 4$ then for some $C > 0$

$$M(re^{i\theta}) \leq C \left(\log \left(\frac{1}{1-r \cos(\theta)} \right) \right)^{\frac{1}{2}} \quad \forall r \in [0, 1), \theta \in [0, 2\pi].$$

(b) If $d \geq 5$ then

$$\sup_{|z| < 1} M(z) < \infty.$$

Proof: We have

$$\begin{aligned} M(z) &= \|z\| \|\mathfrak{F}^{-1}(e^{-t|\xi|^2} [1 - ze^{-t|\xi|^2}]^{-1})\|_{L^2} \\ &= \|z\| \|e^{-t|\xi|^2} [1 - ze^{-t|\xi|^2}]^{-1}\|_{L^2} = |z| \left[\int_{\mathbb{R}^d} \frac{e^{-2t|\xi|^2}}{|1 - ze^{-t|\xi|^2}|^2} d\xi \right]^{\frac{1}{2}}. \end{aligned} \tag{19}$$

From (19) one sees that $M(z)$ is uniformly bounded in the complement of any neighborhood of the point $z = 1$ in the unit disk, so that the issue is to study

the behavior of $M(z)$ when $z \rightarrow 1$. It is easy to verify that, for any $|z| < 1$ with $\operatorname{Re}(z) > 0$, $\xi \in \mathbb{R}^d$,

$$|1 - ze^{-|\xi|^2}| \geq 1 - \operatorname{Re}(z)e^{-|\xi|^2},$$

hence

$$\begin{aligned} (M(z))^2 &= |z|^2 \int_{\mathbb{R}^d} \frac{e^{-2t|\xi|^2}}{|1 - ze^{-t|\xi|^2}|^2} d\xi \leq |z|^2 \int_{\mathbb{R}^d} \frac{e^{-2t|\xi|^2}}{(1 - \operatorname{Re}(z)e^{-t|\xi|^2})^2} d\xi \\ &= \omega_d |z|^2 \int_0^\infty \frac{\rho^{d-1} e^{-2t\rho^2}}{(1 - \operatorname{Re}(z)e^{-t\rho^2})^2} d\rho = \omega_d \frac{|z|^2}{2} \frac{1}{t^{\frac{d}{2}}} \int_1^\infty \frac{1}{s} \frac{(\log(s))^{\frac{d}{2}-1}}{(s - \operatorname{Re}(z))^2} ds \\ &\leq \omega_d \frac{|z|^2}{2} \frac{1}{t^{\frac{d}{2}}} \int_1^\infty \frac{(\log(s))^{\frac{d}{2}-1}}{(s - \operatorname{Re}(z))^2} ds = \omega_d \frac{|z|^2}{2} \frac{1}{t^{\frac{d}{2}}} J_{\frac{d}{2}}(\operatorname{Re}(z)), \end{aligned} \quad (20)$$

where ω_d is the $d-1$ -dimensional measure of the unit sphere in \mathbb{R}^d , and J_p is defined by (18). The result follows from (20) and from the estimates in Lemma 4. ■

3.2 Some estimates on D_t

We recall the Duhamel formula

$$D_t = \int_0^t e^{-s(H_0 + V_-)} V_- e^{-(t-s)H_0} ds. \quad (21)$$

The integral kernel corresponding to the operator D_t is denoted by $D_t(x, y)$.

The condition

$$\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |D_t(x, y)| dx \right)^2 dy < \infty. \quad (22)$$

on the kernel of D_t will be essential to us, and will be used in Lemma 7 and 9. The next lemma gives a sufficient condition for (22) to hold.

Lemma 6 *Assuming $V_- \in K(\mathbb{R}^d)$, $t > 0$ and*

$$\int_0^t \int_{\mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} |V_-(w)| e^{-(s+s')H_0}(w, w') |V_-(w')| dw' ds' dw ds < \infty, \quad (23)$$

we have (22).

Proof: By the Duhamel formula (21) we have

$$D_t(x, y) = \int_0^t \int_{\mathbb{R}^d} e^{-s(H_0+V_-)}(x, w) V_-(w) e^{-(t-s)H_0}(w, y) dw ds$$

so

$$\begin{aligned} & \int_{\mathbb{R}^d} |D_t(x, y)| dx \\ & \leq \int_0^t \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} e^{-s(H_0+V_-)}(x, w) dx \right) |V_-(w)| e^{-(t-s)H_0}(w, y) dw ds \\ & \leq \left[\sup_{s \in [0, t], w \in \mathbb{R}^d} \int_{\mathbb{R}^d} e^{-s(H_0+V_-)}(x, w) dx \right] \int_0^t \int_{\mathbb{R}^d} |V_-(w)| e^{-sH_0}(w, y) dw ds, \\ & = \sup_{s \in [0, t]} \|e^{-s(H_0+V_-)}\|_{L^\infty, L^\infty} \int_0^t \int_{\mathbb{R}^d} |V_-(w)| e^{-sH_0}(w, y) dw ds, \end{aligned}$$

which implies

$$\begin{aligned} & \left(\int_{\mathbb{R}^d} |D_t(x, y)| dx \right)^2 \leq \sup_{s \in [0, t]} \|e^{-s(H_0+V_-)}\|_{L^\infty, L^\infty}^2 \\ & \times \int_0^t \int_{\mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} |V_-(w)| e^{-sH_0}(w, y) |V_-(w')| e^{-s'H_0}(w', y) dw' ds' dw ds, \end{aligned}$$

hence

$$\begin{aligned} & \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |D_t(x, y)| dx \right)^2 dy \leq \sup_{s \in [0, t]} \|e^{-s(H_0+V_-)}\|_{L^\infty, L^\infty}^2 \quad (24) \\ & \times \int_0^t \int_{\mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} |V_-(w)| e^{-(s+s')H_0}(w, w') |V_-(w')| dw' ds' dw ds. \end{aligned}$$

and the finiteness of the right-hand side of (24) follows from Lemma 2 and from the assumption (23). ■

Lemma 7 *If $V \in K(\mathbb{R}^d)$ and (22) holds for $t > 0$ sufficiently small, then D_t is Hilbert-Schmidt for $t > 0$ sufficiently small.*

Proof: By the identity

$$D_t = e^{-\frac{t}{2}(H_0+V_-)} D_{\frac{t}{2}} + D_{\frac{t}{2}} e^{-\frac{t}{2}H_0},$$

we have

$$\begin{aligned}\|D_t\|_{HS} &\leq \|e^{-\frac{t}{2}(H_0+V_-)}D_{\frac{t}{2}}\|_{HS} + \|D_{\frac{t}{2}}e^{-\frac{t}{2}H_0}\|_{HS} \\ &= \|e^{-\frac{t}{2}(H_0+V_-)}D_{\frac{t}{2}}\|_{HS} + \|e^{-\frac{t}{2}H_0}D_{\frac{t}{2}}\|_{HS}.\end{aligned}\quad (25)$$

Since

$$\begin{aligned}\|e^{-\frac{t}{2}(H_0+V_-)}D_{\frac{t}{2}}\|_{HS}^2 &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} e^{-\frac{t}{2}(H_0+V_-)}(x, u) D_{\frac{t}{2}}(u, y) du \right)^2 dx dy, \\ \|e^{-\frac{t}{2}H_0}D_{\frac{t}{2}}\|_{HS}^2 &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} e^{-\frac{t}{2}H_0}(x, u) D_{\frac{t}{2}}(u, y) du \right)^2 dx dy,\end{aligned}$$

and since

$$e^{-\frac{t}{2}H_0}(x, u) \leq e^{-\frac{t}{2}(H_0+V_-)}(x, u),$$

we have

$$\|e^{-\frac{t}{2}H_0}D_{\frac{t}{2}}\|_{HS} \leq \|e^{-\frac{t}{2}(H_0+V_-)}D_{\frac{t}{2}}\|_{HS}.\quad (26)$$

From (26) we have,

$$\begin{aligned}&\|e^{-\frac{t}{2}(H_0+V_-)}D_{\frac{t}{2}}\|_{HS}^2 \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{-\frac{t}{2}(H_0+V_-)}(x, u) D_{\frac{t}{2}}(u, y) e^{-\frac{t}{2}(H_0+V_-)}(x, u') D_{\frac{t}{2}}(u', y) du du' dx dy \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{-t(H_0+V_-)}(u, u') D_{\frac{t}{2}}(u, y) D_{\frac{t}{2}}(u', y) du du' dy \\ &\leq \left[\sup_{x, y \in \mathbb{R}^d} e^{-t(H_0+V_-)}(x, y) \right] \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} D_{\frac{t}{2}}(u, y) du \right)^2 dy \\ &\leq \|e^{-t(H_0+V_-)}\|_{L^1, L^\infty} \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} D_{\frac{t}{2}}(u, y) du \right)^2 dy < \infty,\end{aligned}\quad (27)$$

where the finiteness of the two terms of the product on the right-hand side follows from Lemmas 2 and 6.

The result follows from (25), (26) and (27). ■

We now show that the condition (23) (which in turn, by Lemma 6, implies the condition (22) which we need) is implied by the explicit conditions on V_- given in Theorem 1.

Lemma 8 *Assume that V_- satisfies (2) for some $c > 0$.*

- (i) *If $d = 4$ and V_- also satisfies (3) then (23) holds.*
- (ii) *If $d \geq 5$ and V_- also satisfies (4) then (23) holds.*

Proof: We assume that V_- satisfies (2) for some $c = c_0$, and note that this implies that it satisfies (2) for *all* $c \geq c_0$.

We have

$$\begin{aligned}
& \int_0^t \int_{\mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} |V_-(w)| e^{-(s+s')H_0}(w, w') |V_-(w')| dw' ds' dw ds \quad (28) \\
&= \int_0^t \int_0^u \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |V_-(w)| e^{-uH_0}(w, w') |V_-(w')| dw dw' ds du \\
&+ \int_t^{2t} \int_{u-t}^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |V_-(w)| e^{-uH_0}(w, w') |V_-(w')| dw dw' ds du = I_1 + I_2,
\end{aligned}$$

where

$$\begin{aligned}
I_1 &= \int_0^t u \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |V_-(w)| e^{-uH_0}(w, w') |V_-(w')| dw dw' du, \\
I_2 &= \int_0^t u \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |V_-(w)| e^{-(2t-u)H_0}(w, w') |V_-(w')| dw dw' du.
\end{aligned}$$

For I_2 we have

$$\begin{aligned}
I_2 &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |V_-(w)| |V_-(w')| \left(\int_0^t u (4\pi(2t-u))^{-\frac{d}{2}} e^{-\frac{|w-w'|^2}{4(2t-u)}} du \right) dw dw' \\
&\leq t^2 (4\pi t)^{-\frac{d}{2}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{-\frac{|w-w'|^2}{8t}} |V_-(w)| |V_-(w')| dw dw',
\end{aligned}$$

which is finite for $t > 0$ sufficiently small due to the assumption (2).

We are left with showing that I_1 is finite under the stated conditions. We have

$$I_1 = (4\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |V_-(w)| |V_-(w')| \left(\int_0^t u^{1-\frac{d}{2}} e^{-\frac{|w-w'|^2}{4u}} du \right) dw dw'.$$

and making the substitution $v = \frac{a}{u}$, we estimate

$$\int_0^t u^{1-\frac{d}{2}} e^{-\frac{a}{u}} du = a^{2-\frac{d}{2}} \int_{\frac{a}{t}}^{\infty} v^{\frac{d}{2}-3} e^{-v} dv \leq a^{2-\frac{d}{2}} e^{-\frac{a}{2t}} \int_{\frac{a}{t}}^{\infty} v^{\frac{d}{2}-3} e^{-\frac{v}{2}} dv. \quad (29)$$

If $d \geq 5$ then

$$\int_{\frac{a}{t}}^{\infty} v^{\frac{d}{2}-3} e^{-\frac{v}{2}} dv \leq \int_0^{\infty} v^{\frac{d}{2}-3} e^{-\frac{v}{2}} < \infty,$$

so that, putting $a = \frac{|w-w'|^2}{4}$ in (29), we have

$$\int_0^t u^{1-\frac{d}{2}} e^{-\frac{|w-w'|^2}{4u}} du \leq C \frac{e^{-\frac{|w-w'|^2}{8t}}}{|w-w'|^{d-4}},$$

where C is independent of w, w' , so that, by (29),

$$\begin{aligned} I_1 &\leq C \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{e^{-\frac{|w-w'|^2}{8t}}}{|w-w'|^{d-4}} |V_-(w)| |V_-(w')| dw dw' \\ &\leq C \int \int_{|w-w'| \geq 1} e^{-\frac{|w-w'|^2}{8t}} |V_-(w)| |V_-(w')| dw dw' \\ &\quad + C \int \int_{|w-w'| \leq 1} \frac{1}{|w-w'|^{d-4}} |V_-(w)| |V_-(w')| dw dw' \end{aligned}$$

and both of the last two integrals are finite, the first (for sufficiently small $t > 0$) by (2) and the second by (4), so that, in the case $d \geq 5$, (4) we have that I_1 is finite.

To treat the case $d = 4$ we note that, using L'Hôpital's rule, we have

$$\lim_{\alpha \rightarrow 0^+} \frac{1}{\log(\frac{1}{\alpha})} \int_{\alpha}^{\infty} v^{-1} e^{-\frac{v}{2}} dv = 1.$$

We can therefore choose $0 < \alpha_0 < 1$ so that

$$0 < \alpha \leq \alpha_0 \Rightarrow \int_{\alpha}^{\infty} v^{-1} e^{-\frac{v}{2}} dv \leq 2 \log(\alpha^{-1})$$

and then we also have

$$\alpha > \alpha_0 \Rightarrow \int_{\alpha}^{\infty} v^{-1} e^{-\frac{v}{2}} dv \leq \int_{\alpha_0}^{\infty} v^{-1} e^{-\frac{v}{2}} dv \leq 2 \log(\alpha_0^{-1}).$$

Therefore, using (29),

$$0 < a \leq \alpha_0 t \Rightarrow \int_0^t u^{-1} e^{-\frac{a}{u}} du = e^{-\frac{a}{2t}} \int_{\frac{a}{t}}^{\infty} v^{-1} e^{-\frac{v}{2}} dv \leq 2 e^{-\frac{a}{2t}} \log\left(\frac{t}{a}\right),$$

$$a > \alpha_0 t \Rightarrow \int_0^t u^{-1} e^{-\frac{a}{u}} du = e^{-\frac{a}{t}} \int_{\frac{a}{t}}^{\infty} v^{-1} e^{-\frac{v}{2}} dv \leq 2e^{-\frac{a}{t}} \log(\alpha_0^{-1})$$

and setting $a = \frac{|w-w'|^2}{4}$

$$0 < |w - w'| \leq 2\sqrt{\alpha_0 t} \Rightarrow \int_0^t u^{-1} e^{-\frac{|w-w'|^2}{4u}} du \leq 2e^{-\frac{|w-w'|^2}{8t}} \log\left(\frac{4t}{|w-w'|^2}\right),$$

$$|w - w'| > 2\sqrt{\alpha_0 t} \Rightarrow \int_0^t u^{-1} e^{-\frac{|w-w'|^2}{4u}} du \leq 2e^{-\frac{|w-w'|^2}{8t}} \log(\alpha_0^{-1}).$$

Hence, using (29),

$$\begin{aligned} I_1 &\leq 2\log(\alpha_0^{-1})(4\pi)^{-\frac{d}{2}} \iint_{|w-w'| > 2\sqrt{\alpha_0 t}} e^{-\frac{|w-w'|^2}{8t}} |V_-(w)| |V_-(w')| dw dw' \\ &\quad + 4(4\pi)^{-\frac{d}{2}} \iint_{|w-w'| < 2\sqrt{\alpha_0 t}} \log\left(\frac{2\sqrt{t}}{|w-w'|}\right) |V_-(w)| |V_-(w')| dw dw'. \end{aligned}$$

The first integral above is finite for t sufficiently small, due to (2). For the second integral we have

$$\begin{aligned} &\iint_{|w-w'| < 2\sqrt{\alpha_0 t}} \log\left(\frac{2\sqrt{t}}{|w-w'|}\right) |V_-(w)| |V_-(w')| dw dw' \\ &= \iint_{|w-w'| < 2\sqrt{\alpha_0 t}} \log\left(\frac{1}{|w-w'|}\right) |V_-(w)| |V_-(w')| dw dw' \\ &\quad + \log(2\sqrt{t}) \iint_{|w-w'| < 2\sqrt{\alpha_0 t}} |V_-(w)| |V_-(w')| dw dw', \end{aligned}$$

and finiteness of the above two integrals for $t > 0$ sufficiently small follows from (3) and (2), respectively. We have thus shown that I_1 is finite when $d = 4$. ■

3.3 Hilbert-Schmidt norm bound for the composition of $G(z)$ and D_t

Lemma 9 Assume $V_- \in K(\mathbb{R}^d)$ and that (22) holds. Then $G(z)D_t$ is Hilbert-Schmidt, and

$$\|G(z)D_t\|_{HS} \leq M(z) \left[\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |D_t(u, y)| du \right)^2 dy \right]^{\frac{1}{2}},$$

Proof: We have

$$[G(z)D_t](x, y) = \int_{\mathbb{R}^d} g_z(x - u)D_t(u, y)du,$$

hence

$$([G(z)D_t](x, y))^2 = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g_z(x - u)D_t(u, y)g_z(x - v)D_t(v, y)dudv$$

and thus

$$\begin{aligned} \|G(z)D_t\|_{HS}^2 &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} ([G(z)D_t](x, y))^2 dx dy \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} g_z(x - u)g_z(x - v)dx \right) \left(\int_{\mathbb{R}^d} D_t(u, y)D_t(v, y)dy \right) dudv \\ &\leq \left[\sup_{u, v \in \mathbb{R}^d} \left(\int_{\mathbb{R}^d} |g_z(x - u)g_z(x - v)|dx \right) \right] \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |D_t(u, y)D_t(v, y)|dy dudv. \\ &= \left[\sup_{u, v \in \mathbb{R}^d} \left(\int_{\mathbb{R}^d} |g_z(x - u)g_z(x - v)|dx \right) \right] \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |D_t(u, y)|du \right)^2 dy. \end{aligned} \quad (30)$$

We also have

$$\begin{aligned} &\sup_{u, v \in \mathbb{R}^d} \int_{\mathbb{R}^d} |g_z(x - u)g_z(x - v)|dx \\ &\leq \sup_{u, v \in \mathbb{R}^d} \left(\int_{\mathbb{R}^d} |g_z(x - u)|^2 dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^d} |g_z(x - v)|^2 dx \right)^{\frac{1}{2}} = \|g_z\|_{L^2}^2 = (M(z))^2. \end{aligned} \quad (31)$$

The result follows from (30) and (31). ■

3.4 Bounding the Jensen integral

Lemma 10 Assume $V_- \in K(\mathbb{R}^d)$ and (22) holds. Then we have, for all $|z| < 1$,

$$\|F(z)\|_{HS} \leq |z|[C_1 + C_2 M(z)].$$

where

$$C_1 = \|D_t\|_{HS}, \quad C_2 = \left[\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |D_t(u, y)|du \right)^2 dy \right]^{\frac{1}{2}}.$$

Proof: Returning to (15) and using Lemmas 7 and 9 we get the result. ■

We are now ready for

Proof of Theorem 1: By the above lemma and by (11) we have

$$\log(|h(z)|) \leq \frac{1}{2} \|F(z)\|_{HS}^2 \leq \frac{1}{2} |z|^2 [C_1 + C_2 M(z)]^2. \quad (32)$$

In the case $d = 4$, (32) and Lemma 5(a) give us

$$\log(|h(re^{i\theta})|) \leq \frac{1}{2} \left[C_1 + C_2 C \left(\log \left(\frac{1}{1 - r \cos(\theta)} \right) \right)^{\frac{1}{2}} \right]^2 \leq C' \log \left(\frac{1}{1 - r \cos(\theta)} \right),$$

so that

$$\limsup_{r \rightarrow 1^-} \int_0^{2\pi} \log(|h(re^{i\theta})|) d\theta \leq C' \int_0^{2\pi} \log \left(\frac{1}{1 - \cos(\theta)} \right) d\theta < \infty,$$

which again, by Theorem 2, gives us (1).

In the case $d \geq 5$, Lemma 5(c) tells us that

$$\log(|h(re^{i\theta})|) \leq C \quad \forall r \in [0, 1), \quad \theta \in [0, 2\pi]$$

so that

$$\limsup_{r \rightarrow 1^-} \int_0^{2\pi} \log(|h(re^{i\theta})|) d\theta < \infty$$

and again Theorem 2 gives us (1).

We now prove the corollaries.

Proof of Corollary 1: We use Young's inequality:

$$\|f * g\|_{L^r} \leq C \|f\|_{L^p} \|g\|_{L^q}$$

valid for $p, q, r \geq 1$ with $\frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{q}$.

Taking $f(x) = |V_-(x)|$, $g(x) = e^{-c|x|^2}$, and

$$p \in [1, 2], \quad q = \frac{p}{2(p-1)}, \quad r = \frac{p}{p-1} \quad (33)$$

(note that since $p \leq 2$ we have $q \geq 1$), we have that if $V_- \in L^p(\mathbb{R}^d)$ then $f * g \in L^{\frac{p}{p-1}}(\mathbb{R}^d)$. Thus, using Hölder's inequality we have

$$\begin{aligned} & \iint_{\mathbb{R}^d} e^{-c|w-w'|^2} |V_-(w)| |V_-(w')| dw' dw \\ &= \int_{\mathbb{R}^d} |V_-(w)| (f * g)(w) dw \leq \|V_-\|_{L^p} \|f * g\|_{L^{\frac{p}{p-1}}} < \infty. \end{aligned}$$

Thus (2) holds whenever $V_- \in L^p(\mathbb{R}^d)$, $p \in [1, 2]$.

To verify (3) (for the case $d = 4$) we take $f = |V_-|$, $g(x) = \log(\frac{1}{|x|})\chi_{B_1}$, where χ_{B_1} is the characteristic function of the unit ball. and p, q, r according to (33). We assume $V_- \in L^p$, and note that $g \in L^q(\mathbb{R}^4)$. Hence by Young's inequality we have $f * g \in L^{\frac{p}{p-1}}(\mathbb{R}^4)$. Therefore, using Hölder's inequality, we have

$$\begin{aligned} & \iint_{|w-w'|<1} \log\left(\frac{1}{|w-w'|}\right) |V_-(w)| |V_-(w')| dw' dw \\ &= \int_{\mathbb{R}^4} |V_-(w)| (f * g)(w) dw \leq \|V_-\|_{L^p} \|f * g\|_{L^{\frac{p}{p-1}}} < \infty, \end{aligned}$$

so that (3) is satisfied for any $V_- \in L^p(\mathbb{R}^4)$, $p \in (1, 2]$.

To verify (4) (for the case $d \geq 5$) we take $f = |V_-|$, $g(x) = \frac{1}{|x|^{d-4}}\chi_{B_1}$, and p, q, r defined by (33). Note that in order to have $g \in L^q(\mathbb{R}^d)$ we need the condition: $q(d-4) < d$, that is $p > \frac{2d}{d+4}$, and in that case we get, assuming $V_- \in L^p$, $\|f * g\|_{L^{\frac{p}{p-1}}} < \infty$, and thus

$$\begin{aligned} & \iint_{|w-w'|<1} \frac{|V_-(w)| |V_-(w')|}{|w-w'|^{d-4}} dw dw' \\ &= \int_{\mathbb{R}^d} |V_-(w)| (f * g)(w) dw \leq \|V_-\|_{L^p} \|f * g\|_{L^{\frac{p}{p-1}}} < \infty. \end{aligned}$$

Thus we have shown that (4) holds when $p > \frac{2d}{d+4}$. To show that it also holds when $p = \frac{2d}{d+4}$, we need to use the Hardy-Littlewood-Sobolev inequality (see e.g. [4], Theorem 4.3), which says that

$$\left| \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{f(w)g(w')}{|w-w'|^\lambda} dw dw' \right| \leq C \|f\|_{L^p} \|g\|_{L^r},$$

where $p, r > 1$, $0 < \lambda < d$, and $\frac{1}{p} + \frac{1}{r} = 2 - \frac{\lambda}{d}$. We take $p = r = \frac{2d}{d+4}$ (since $d \geq 5$ we have $p, r > 1$), $\lambda = d - 4$, $f = g = |V_-|$, to obtain

$$\iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|V_-(w)| |V_-(w')|}{|w-w'|^{d-4}} dw dw' < \infty,$$

which is even stronger than (4).

Proof of Corollary 2: Since Corollary 1 contains the result for the case $d = 4$, we can assume that $d \geq 5$. Since $V_- \in L^1(\mathbb{R}^d)$, we can write

$$\iint_{|w-w'|<1} \frac{|V_-(w)| |V_-(w')|}{|w-w'|^{d-4}} dw dw' \leq \|V\|_{L^1} \sup_{w \in \mathbb{R}^d} \int_{|w-w'|<1} \frac{|V_-(w')|}{|w-w'|^{d-4}} dw'.$$

Thus we only need to show that the supremum on the right-hand side is finite. Since $V_- \in K(\mathbb{R}^d)$, we have from (13) that there exists $\alpha < 1$ such that

$$\sup_{w \in \mathbb{R}^d} \int_{|w-w'| \leq \alpha} \frac{|V(w')|}{|w-w'|^{d-2}} dw' = \eta < \infty.$$

Thus

$$\begin{aligned} & \sup_{w \in \mathbb{R}^d} \int_{|w-w'| < 1} \frac{|V_-(w')|}{|w-w'|^{d-4}} dw' \\ & \leq \sup_{w \in \mathbb{R}^d} \int_{|w-w'| \leq \alpha} \frac{|V_-(w')|}{|w-w'|^{d-4}} dw' + \sup_{w \in \mathbb{R}^d} \int_{\alpha < |w-w'| \leq 1} \frac{|V_-(w')|}{|w-w'|^{d-4}} dw' \\ & \leq \sup_{w \in \mathbb{R}^d} \int_{|w-w'| \leq \alpha} \frac{|V_-(w')|}{|w-w'|^{d-2}} dw' + \frac{1}{\alpha^{d-4}} \sup_{w \in \mathbb{R}^d} \int_{\alpha < |w-w'| \leq 1} |V_-(w')| dw' \\ & \leq \eta + \frac{1}{\alpha^{d-4}} \|V_-\|_{L^1}, \end{aligned}$$

and we have proved the required finiteness.

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